Journal of Algebra, Number Theory: Advances and Applications
Volume 5, Number 2, 2011, Pages 41-59

ON MONOIDAL EQUIVALENCES AND ANN-EQUIVALENCES

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Abstract

In this paper, we prove that a monoidal functor (an Ann-functor) F is a monoidal equivalence (resp., an Ann-equivalence) iff F is a categorical equivalence. Then, we introduce a general method to make the constraints of a monoidal category and of an Ann-category be strict.

1. Introduction

Monoidal categories appear in every mathematical fields. They are the most simple example of categories with an operation and it is also a categorification of the notion of monoid. Some more complexed algebraic structures such as groups, abelian groups, and rings have been categorified by the notions of Gr-categories, braided categories, Pic-categories, and Ann-categories. To make the uses of these notions conveniently, we need to make each axiomatics more simple (so-called the stricting of the constraints). There are many different proofs for stricting

2010 Mathematics Subject Classification: 18D10.

Keywords and phrases: monoidal category, Ann-category, strict constraint.

Received April 9, 2011

the constraints of monoidal categories (see [1], [9]). Similar results have been stated for Gr-categories [6], for Ann-categories [5]. Monoidal categories with a strict associativity constraint have been used to construct a braided categories (see [1]), or to study unit constraints (see [2]).

The main content of this paper is to introduce characters of monoidal equivalences and Ann-equivalences, and to apply them to the problem of stricting the constraints of monoidal categories and of Ann-categories with the same technique. It is known that if M is an A-module, then $Hom_A(A, M) \cong M$. This result can be extended for Ann-categories, where module homomorphisms are replaced by μ-functors. If the notion of μ-functors is replaced by a weaker notion, we shall obtain an equivalence between a monoidal category and a strict one. The notion of μ-functor was first introduced in [5] (in Vietnamese). The proof that any Ann-category is equivalent to an almost strict one (Theorem 3.11), is firstly a full and exact modification of [5], and secondly includes the proof that any monoidal category is equivalent to a strict one (Theorem 3.6).

Fundamental notions on monoidal categories and Ann-categories can be found in [3, 4, 7, 8]. Hereafter, for any objects A and B, for convenience, let us denote AB instead of $A \otimes B$. However, for morphisms, we still denote $f \otimes g$ to avoid confusion with a composition.

2. Ann-Equivalences Between Ann-Categories

2.1. Monoidal equivalences

A monoidal category $(\mathbf{C}, \otimes, I, a, l, r)$ is a category $\mathbf{C}$, which is equipped with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$; with an object I, called *the unit object* and with isomorphisms, which are, respectively, called *the associativity constraint*, *the left and right unit constraint*

$$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$

$$l_A : I \otimes A \to A,\ r_A : A \otimes I \to A,$$

satisfying the coherence conditions

$$(a_{A,B,C} \otimes id_D) \cdot a_{A,B\otimes C,D} \cdot (id_A \otimes a_{B,C,D}) = a_{A\otimes B,C,D} \cdot a_{A,B,C\otimes D},$$

$$id_A \otimes l_B = (r_A \otimes id_B) \cdot a_{A,I,B}.$$

A monoidal category is *strict*, if the constraints a, l, r are all identities.

Let $\mathbf{C} = (\mathbf{C}, \otimes, I, a, l, r)$ and $\mathbf{D} = (\mathbf{D}, \otimes, I', a', l', r')$ be monoidal categories, *a monoidal functor* from $\mathbf{C}$ to $\mathbf{D}$ is a triple $(F, \widetilde{F}, \widehat{F})$, where $F : \mathbf{C} \to \mathbf{D}$ is a functor, a natural isomorphism

$$\widetilde{F}_{A,B} : F(A \otimes B) \to FA \otimes FB,$$

and an isomorphism $\widehat{F} : FI \to I'$, satisfying the following coherence conditions:

$$(\widetilde{F}_{A,B} \otimes FC) \cdot \widetilde{F}_{AB,C} \cdot F(a_{A,B,C}) = a'_{FA,FB,FC} \cdot (FA \otimes \widetilde{F}_{B,C}) \cdot \widetilde{F}_{A,BC}, \tag{1.1}$$

$$F(r_A) = \widetilde{F}_{I,A} \cdot (id \otimes \widehat{F}) \cdot r'_{FA}, \tag{1.2}$$

$$F(l_A) = \widetilde{F}_{A,I} \cdot (\widehat{F} \otimes id) \cdot l'_{FA}. \tag{1.3}$$

A monoidal natural transformation $\alpha : (F, \widetilde{F}, \widehat{F}) \to (G, \widetilde{G}, \widehat{G})$ between monoidal functors from $\mathbf{C}$ to $\mathbf{C}'$ is a natural transformation $\alpha : F \to G$, such that

$$\widehat{F} = \widehat{G} \cdot \alpha_I, \tag{1.4}$$

and for all pairs $(\mathbf{A}, \mathbf{B})$ of objects in $\mathbf{C}$

$$(\alpha_A \otimes \alpha_B) \cdot \widetilde{F}_{A,B} = \widetilde{G}_{A,B} \cdot \alpha_{AB}. \tag{1.5}$$

A monoidal functor $F : \mathbf{C} \to \mathbf{D}$ is *a monoidal equivalence*, if there exists a monoidal functor $G : \mathbf{D} \to \mathbf{C}$ together with natural monoidal isomorphisms $\alpha : G \cdot F \to \mathrm{id}_{\mathbf{C}}$ and $\beta : F \cdot G \to \mathrm{id}_{\mathbf{D}}$. Two monoidal categories are *monoidal equivalent*, if there exists a monoidal equivalence between them.

We shall first prove a simple character of monoidal equivalences.

Theorem 2.1. *A monoidal functor* $F : \mathbf{C} \to \mathbf{D}$ *is a monoidal equivalence iff F is a categorical equivalence.*

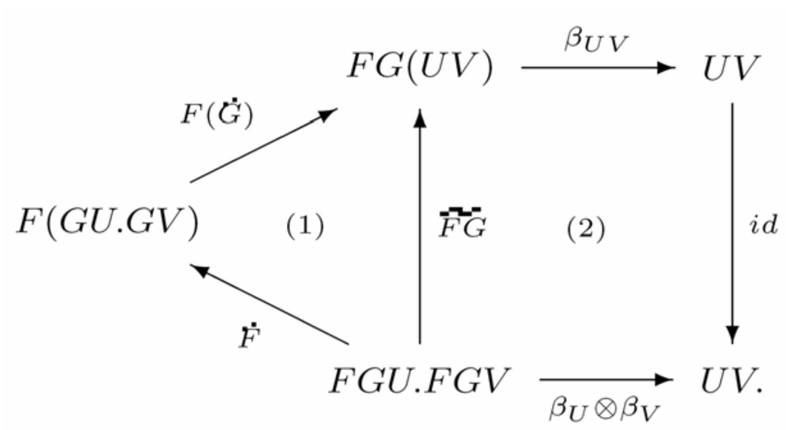
Proof. Let $F : \mathbf{C} \to \mathbf{D}$ be a monoidal functor with a natural isomorphism

$$\widetilde{F}_{X,Y} : FX \otimes FY \to F(X \otimes Y).$$

Since F is a categorical equivalence, there exists a functor $G : \mathbf{D} \to \mathbf{C}$ and morphisms $\alpha : GF \to \mathrm{id}_{\mathbf{C}}$ and $\beta : FG \to \mathrm{id}_{\mathbf{D}}$. Moreover, we can choose α, β such that the quadruple (F, G, α, β) satisfies $F(\alpha_A) = \beta_{FA}$; $G(\beta_B) = \alpha_{GB}$, for all objects $A \in \mathbf{C}$, $B \in \mathbf{D}$. The natural isomorphism

$$\widetilde{G}_{U,V} : GU \otimes GV \to G(U \otimes V),$$

for $U, V \in \mathbf{D}$ is defined by the commutative perimeter, $\widetilde{FG}$ is defined by the commutative region (1) of the following diagram:



It follows that the region (2) commutes, so β is an $\otimes$-morphism. Finally, α is a morphism since β is a morphism. □

2.2. Ann-equivalences

A *Pic-category* is a Gr-category together with a commutativity constraint *c*, which is compatible with the associativity one. In other words, a Pic-category is a symmetric monoidal category in which every object is invertible and every morphism is an isomorphism. It is considered as a categorification of the abelian group structure. The notion of Ann-categories is constructed as a categorification of the ring structure. An *Ann-category* consists of

(i) a category $\mathbf{A}$ together with two bifunctors $\oplus, \otimes : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$;

(ii) a fixed object $0 \in \mathbf{A}$ together with natural isomorphisms a^+, c, g, d such that $(\mathbf{A}, \oplus, a^+, c, (0, g, d))$ is a Picard category (or a Pic-category);

(iii) a fixed object $I \in \mathbf{A}$ together with natural isomorphisms a, l, r such that $(\mathbf{A}, \otimes, a, (I, l, r))$ is a monoidal category;

(iv) the distributive natural isomorphisms $\mathfrak{L}, \mathfrak{R}$

$$\mathfrak{L}_{A, X, Y} : A \otimes (X \oplus Y) \to (A \otimes X) \oplus (A \otimes Y),$$

$$\mathfrak{R}_{A, X, Y} : (X \oplus Y) \otimes A \to (X \otimes A) \oplus (Y \otimes A),$$

satisfy the coherence conditions (see [4] for more detail).

Let $\mathbf{A}, \mathbf{B}$ be Ann-categories. An *Ann-functor* from $\mathbf{A}$ to $\mathbf{B}$ is a triple $(F, \breve{F}, \widetilde{F})$, where $(F, \breve{F})$ is a symmetric monoidal $\oplus$- functor and $(F, \widetilde{F})$ is a monoidal $\otimes$- functor such that the two following diagrams commute:

$$\begin{array}{ccccc} F(X(Y \oplus Z)) & \xrightarrow{\dot{F}} & FXF(Y \oplus Z) & \xrightarrow{id \otimes \breve{F}} & FX(FY \oplus FZ) \\ \downarrow{\scriptstyle F(\mathfrak{L})} & & & & \downarrow{\scriptstyle \mathfrak{L}'} \\ F(XY \oplus XZ) & \xrightarrow{\breve{F}} & F(XY) \oplus F(XZ) & \xrightarrow{\hat{F} \oplus \dot{F}} & FXFY \oplus FXFZ, \end{array} \tag{2.1}$$

$$\begin{array}{ccccc}
F((X\oplus Y)Z) & \xrightarrow{\dot{F}} & F(X\oplus Y)FZ & \xrightarrow{\breve{F}\otimes \mathrm{id}} & (FX\oplus FY)FZ \\
\Big\downarrow {\scriptstyle F(\mathfrak{R})} & & & & \Big\downarrow {\scriptstyle \mathfrak{R}'} \\
F(XZ\oplus YZ) & \xrightarrow{\breve{F}} & F(XZ)\oplus F(YZ) & \xrightarrow{\hat{F}\oplus \dot{F}} & FXFZ\oplus FYFZ.
\end{array} \tag{2.2}$$

Let F, G be Ann-functors. A morphism $\varphi : F \to G$ is an *Ann-morphism*, if it is an $\oplus$- morphism, as well as an $\otimes$- morphism, i.e., the following diagrams commute:

$$\begin{array}{ccc}
F(A\oplus B) & \xrightarrow{\breve{F}} & FA\oplus FB \\
{\scriptstyle \varphi_{A\oplus B}}\Big\downarrow & & \Big\downarrow {\scriptstyle \varphi_A\oplus\varphi_B} \\
G(A\oplus B) & \xrightarrow{\breve{G}} & GA\oplus GB
\end{array}
\qquad
\begin{array}{ccc}
F(AB) & \xrightarrow{\dot{F}} & FAFB \\
{\scriptstyle \varphi_{AB}}\Big\downarrow & & \Big\downarrow {\scriptstyle \varphi_A\otimes\varphi_B} \\
G(AB) & \xrightarrow{\dot{G}} & GAGB
\end{array}$$

An Ann-functor $F : \mathbf{A} \to \mathbf{B}$ is an *Ann-equivalence*, if there exists another one $G : \mathbf{B} \to \mathbf{A}$ and Ann-isomorphisms $\alpha : GF \cong \mathrm{id}_{\mathbf{A}}$, $\beta : FG \cong \mathrm{id}_{\mathbf{B}}$. Two Ann-categories are *Ann-equivalent*, if there exists an Ann-equivalence between them. The main result of this section is the following theorem:

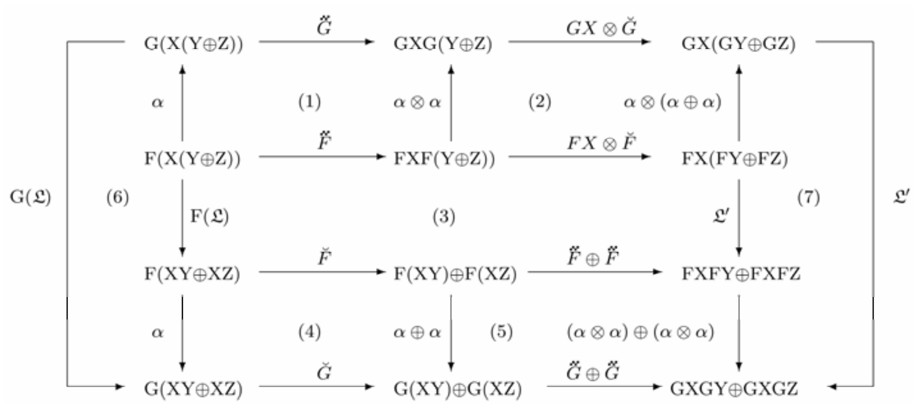
Theorem 2.2. *An Ann-functor* $F : \mathbf{A} \to \mathbf{B}$ *is an Ann-equivalence iff* F *is a categorical equivalence.*

In order to prove this theorem, we first prove the following lemma:

Lemma 2.3. *If the natural equivalence* $\alpha : F \cong G$ *is an* $\oplus$*-morphism, as well as an* $\otimes$*-morphism, and the functor* $F : \mathbf{A} \to \mathbf{B}$ *is compatible with the left distributivity constraints* $\mathfrak{L}$, $\mathfrak{L}'$, *then* G *is also compatible with* $\mathfrak{L}$, $\mathfrak{L}'$. *Similarly, this holds for the right distributivity constraints* $\mathfrak{R}$, $\mathfrak{R}'$.

Proof. In the following diagram, the regions (1) and (5) commute since α is an $\otimes$- morphism; the regions (2) and (4) commute since α is an

$\oplus$- morphism; the region (3) commutes thanks to the compatibility of F with $\mathfrak{L}$, $\mathfrak{L}'$; the region (6) commutes since α is a morphism; and the region (7) commutes thanks to the naturality of $\mathfrak{L}'$.

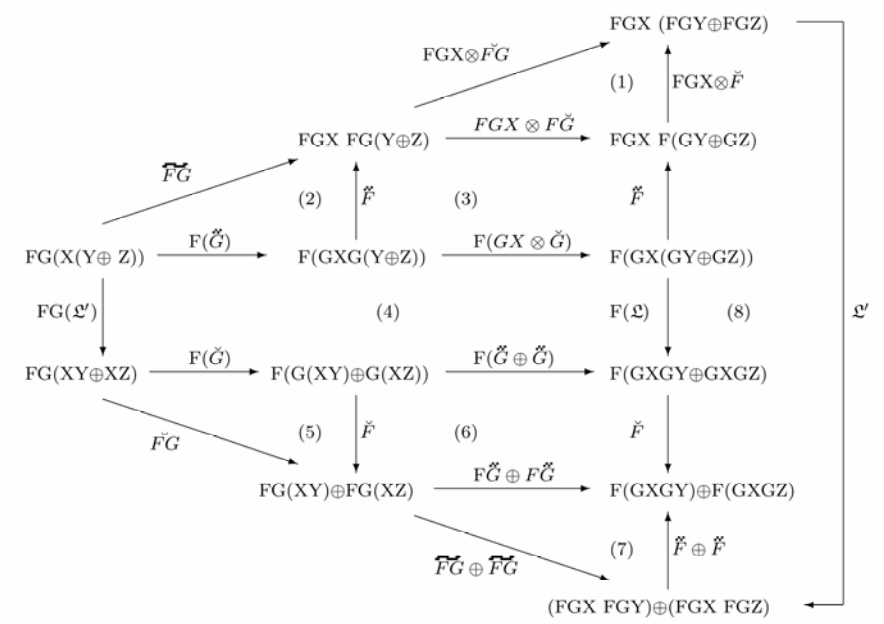


Hence, the perimeter commutes, it implies that G is compatible with $\mathfrak{L}$, $\mathfrak{L}'$. $\square$

Using the above lemma, we prove the following lemma:

Lemma 2.4. *Let* $F : \mathbf{A} \to \mathbf{B}$ *and* $G : \mathbf{B} \to \mathbf{A}$ *be functors. If F is compatible with* $\mathfrak{L}$, $\mathfrak{L}'$ *and the natural isomorphism* $\alpha : FG \cong \mathrm{id}_{\mathbf{B}}$ *is an* $\oplus$- *morphism, as well as an* $\otimes$*-morphism, then G is also compatible with* $\mathfrak{L}'$, $\mathfrak{L}$.

Proof. Consider the following diagram: In this diagram, the regions (1) and (5) commute thanks to the definition of $\breve{FG}$; the regions (2) and (7) commute thanks to the definition of $\widetilde{FG}$; the region (3) commutes thanks to the naturality of $\widetilde{F}$; the region (6) commutes thanks to the naturality of $\breve{F}$; and the region (8) commutes thanks to the compatibility of F with $\mathfrak{L}$, $\mathfrak{L}'$. According to Lemma 2.3, FG is compatible with $\mathfrak{L}'$, $\mathfrak{L}'$, so the perimeter commutes. Therefore, the region (4) commutes; this region is just the image through F of the diagram determining the compatibility of G with $\mathfrak{L}'$, $\mathfrak{L}$.



Since F is faithful, G is compatible with $\mathfrak{L}$, $\mathfrak{L}'$. This completes the proof.

□

The proof of Theorem 2.2. By Theorem 2.1, there exists a functor $G : \mathbf{B} \to \mathbf{A}$ and a natural isomorphism $\widetilde{G}$ such that $(G, \widetilde{G})$ is a monoidal $\otimes$-functor and $\alpha : GF \cong \mathrm{id}_{\mathbf{A}}$, $\beta : FG \cong \mathrm{id}_{\mathbf{B}}$ are $\otimes$-isomorphisms. For the operation $\oplus$, there exists a natural isomorphism $\breve{G}$ such that $(G, \breve{G})$ is an $\oplus$-functor and α, β are $\oplus$-isomorphisms. By Lemma 2.3, the triple $(G, \breve{G}, \widetilde{G})$ is an Ann-functor. Therefore, $(F, \breve{F}, \widetilde{F})$ is an Ann-equivalence. □

3. Almost Strict Ann-Categories

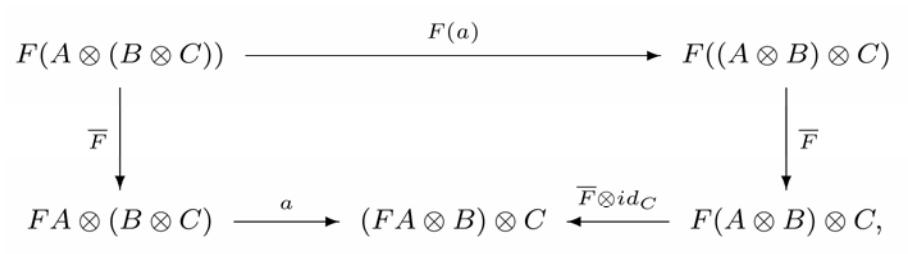
3.1. Strict monoidal categories

The proof of Cayley theorem for groups and the proof of $Hom_R(R, M) \cong M$, for R-modules use the same technique. This suggests us a united technique to make a monoidal category, as well as an Ann-category be strict.

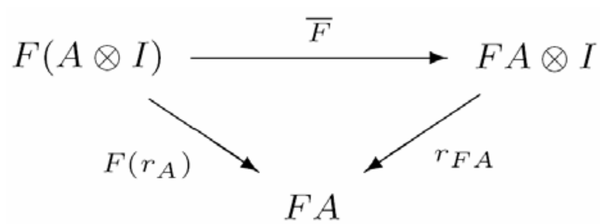
Definition 3.1. An *M-functor* of the monoidal category $\mathbf{C}$ is a pair $(F, \overline{F})$, consisting of a functor $F : \mathbf{C} \to \mathbf{C}$ and a natural isomorphisms

$$\overline{F}_{A, B} : F(A \otimes B) \to FA \otimes B,$$

such that the following diagrams commute:

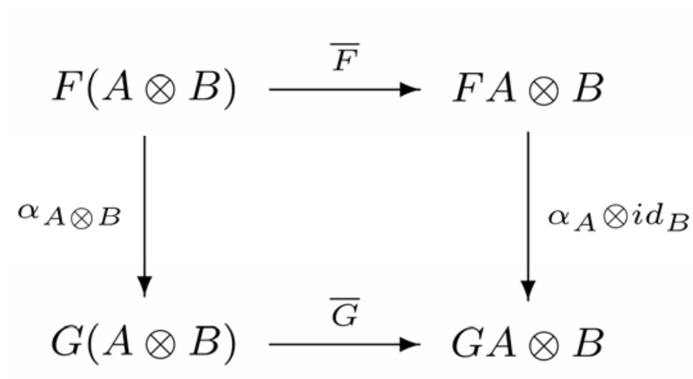


(3.1)



(3.2)

Let $(F, \overline{F})$ and $(G, \overline{G})$ be M-functors of $\mathbf{C}$. An *M-morphism* $\alpha : (F, \overline{F}) \to (G, \overline{G})$ is a morphism $\alpha : F \to G$ such that the following diagram commute:



(3.3)

The composition of two M-morphisms is known as the composition of two usual morphisms. One can verify that the composition of M-morphisms is also an M-morphism.

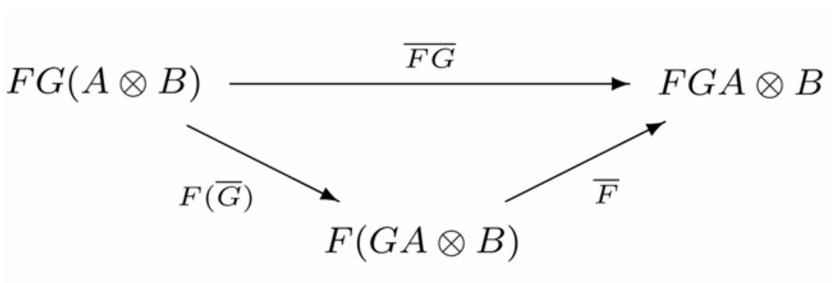
Example. For any object X of $\mathbf{C}$, the pair $(L^X, \overline{L}^X)$ defined by

$$L^X(A) = X \otimes A, \quad L^X(u) = id_X \otimes u, \quad \overline{L}^X_{A,B} = a_{X,A,B},$$

is an M-functor of $\mathbf{C}$. For any pair (X, Y) of objects of $\mathbf{C}$ and a morphism $f : X \to Y$, the morphism $\alpha : L^X \to L^Y$ given by $\alpha_A = f \otimes \mathrm{id}_{\mathbf{A}}$ is a M-morphism of $\mathbf{C}$.

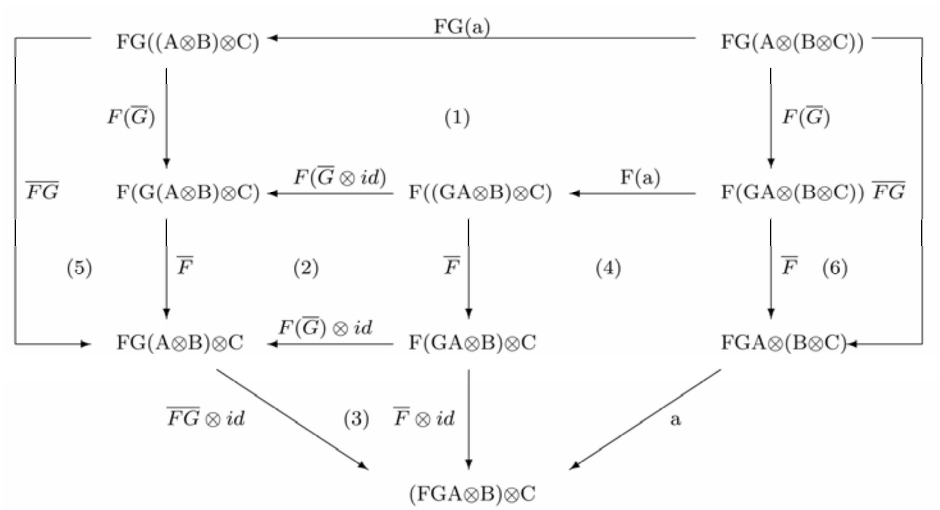
The set of all M-functors and M-morphisms of $\mathbf{C}$ forms a category, denoted by $\mathbf{M}(\mathbf{C})$. We now equip $\mathbf{M}(\mathbf{C})$ with an operation $\otimes$ together with an associativity constraint a^*, a unit constraint (I^*, l^*, r^*) to make it become a monoidal category.

Lemma 3.2. *Let* $(F, \overline{F})$ *and* $(G, \overline{G})$ *be M-functors of* $\mathbf{C}$. *Then, the composition FG is also an M-functor with the natural isomorphism* $\overline{FG}$ *defined by following commutative diagram, for any pair* (A, B) *of objects of* $\mathbf{C}$:



(3.4)

Proof. In the following diagram, the regions (1) and (4) commute thanks to Diagram (3.1) for the M-functors F, G. The region (2) commutes thanks to the naturality of $\overline{F}$. The regions (3), (5), and (6) commute thanks to the definition of the isomorphism $\overline{FG}$. Therefore, the perimeter which is the Diagram (3.1) for $(FG, \overline{FG})$ commutes.



We now prove that $\overline{FG}$ satisfies (3.2). For any object A of $\mathbf{C}$, we have

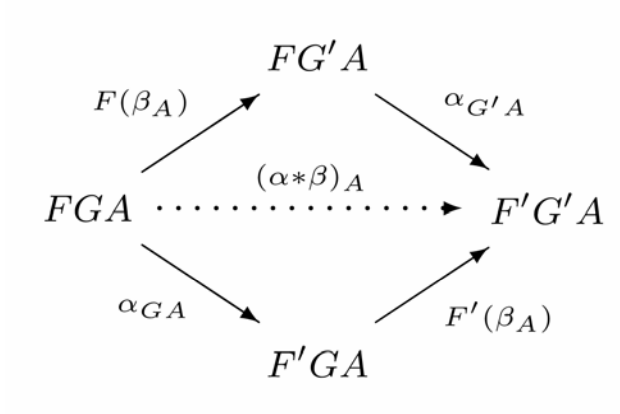
$$r_{FGA} \cdot \overline{FG}_{A,I} \overset{(3.4)}{=} r_{FGA} \cdot (\overline{F}_{GA,I} \cdot F(\overline{G}_{A,I})) \overset{(3.2)}{=} F(r_{GA}) \cdot F(\overline{G}_{A,I}) \overset{(3.2)}{=} FG(r_A).$$

□

Lemma 3.3. *For any pair* $((F, \overline{F}) \xrightarrow{\alpha} (F', \overline{F}'); (G, \overline{G}) \xrightarrow{\beta} (G', \overline{G}'))$ *of M-morphisms of* $\mathcal{C}$, *the morphism*

$$\alpha * \beta : FG \to F'G',$$

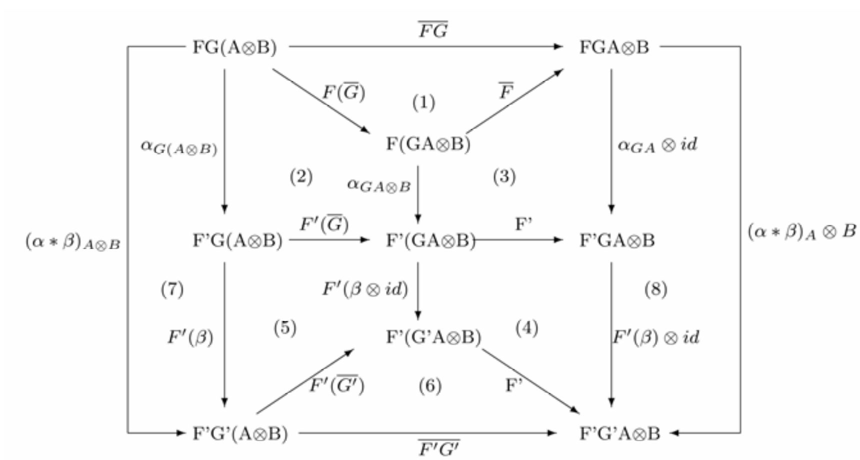
defined by the following commutative diagram:



(3.5)

is an M-morphism from $(FG, \overline{FG})$ *to* $(F'G', \overline{F'G'})$.

Proof. Consider the following diagram. The regions (1) and (6) commute thanks to definitions of the isomorphisms $\overline{FG}$ and $\overline{F'G'}$ (Diagram (3.4)), the regions (3) and (5) commute thanks to definitions of M-morphisms (Diagram (3.3)) α, $\overline{F}'$. The regions (2) and (4) commute thanks to the naturality of the morphisms α, β. The regions (7) and (8) commute thanks to the determination of $\alpha * \beta$ (Diagram (3.5)).



Hence, the perimeter which is the Diagram (3.5) for $\alpha * \beta$ commutes. □

Lemma 3.4. *The category* $\mathbf{M}(\mathbf{C})$ *becomes an* $\otimes$*-category together with the operation* $\otimes$ *defined by*

$$(F,\, \overline{F}) \otimes (G,\, \overline{G}) = (FG,\, \overline{FG}),$$

$$\alpha \otimes \beta = \alpha * \beta : FG \to F'G'.$$

Proof. Thanks to Lemmas 3.2 and 3.3, the tensor product of two M-functors is an M-functor and the tensor product of two M-morphisms is an M-morphism. One can verify that the law $\otimes$ determined above is a tensor operation. Therefore, $\mathbf{M}(\mathbf{C})$ is an $\otimes$-category. □

For M-functors $(F,\, \overline{F}),\, (G,\, \overline{G}),\, (H,\, \overline{H}),$ one can easily prove that $\overline{F(GH)} = \overline{(FG)H}$. So, the identity is an associativity constraint with respect to $\otimes$ on $\mathbf{M}(\mathbf{C})$. The $\otimes$-category $\mathbf{M}(\mathbf{C})$ has unit object $I^* = (Id,\, \overline{Id})$, where Id is an identity functor of $\mathbf{C}$ and $\overline{Id} = id$. Moreover, unit constraints can be chosen to be identities. So, we obtain the following lemma:

Lemma 3.5. *The tensor category* $\mathbf{M}(\mathbf{C})$ *is a strict monoidal category.*

From Theorem 2.1 and Lemma 3.5, we obtain the following result:

Theorem 3.6. *Each monoidal category is monoidal equivalent to a strict one.*

Proof. We now prove the theorem in the following steps:

Step 1. Define a monoidal functor $\Phi : \mathbf{C} \to \mathbf{M}(\mathbf{C})$ as follows:

$$\Phi(A) = (L^A,\, \overline{L}^A),$$

$$\Phi(f)_X = f \otimes id_X : A \otimes X \to B \otimes X,$$

for any objects A, X and any morphism $f : A \to B$ in $\mathcal{C}$. From the above example, $(L^A,\, \overline{L}^A)$ is an M-functor and $\Phi(f)$ is an M-morphism.

Furthermore, one can verify that the triple $(\Phi, \widetilde{\Phi}, \widehat{\Phi})$ is a monoidal functor, where $\widetilde{\Phi}$ and $\widehat{\Phi}$ are isomorphisms defined by

$$\widetilde{\Phi}_{A,B}(X) = (a_{A,B,X})^{-1} : (A \otimes B) \otimes X \to A \otimes (B \otimes X),$$

$$(\widehat{\Phi}_X = l_X) : \Phi(I) \to I^*.$$

Step 2. In this step, we prove the triple $(\Phi, \widetilde{\Phi}, \widehat{\Phi})$ is a monoidal equivalence. In order to do this, we have to exhibit a functor, which is the inverse equivalence of Φ. Consider the functor

$$\Gamma : \mathbf{M(C)} \to \mathbf{C},$$

$$\Gamma(F, \overline{F}) = F(I), \quad \Gamma(\alpha) = \alpha_I : F(I) \to G(I),$$

for any M-functor $(F, \overline{F})$ and any M-morphism $\alpha : (F, \overline{F}) \to (G, \overline{G})$.

Observe that $\Gamma\Phi(f) = \Gamma(\Phi f) = (\Phi f)(I) = f \otimes id_I : A \otimes I \to B \otimes I$ for any morphism $f : A \to B$ in $\mathbf{C}$. Then, we have the natural isomorphism $r : \Gamma\Phi \cong Id_C$, where r is a right unit constraint of $\mathbf{C}$. We now prove that, there exists an isomorphism ρ between $\Phi\Gamma$ and $\mathrm{id}_{\mathbf{M(C)}}$ of $\mathbf{M(C)}$. We have

$$(\Phi\Gamma)(F, \overline{F}) = \Phi(f(I)) = (L^{FI}, \overline{L}^{FI}),$$

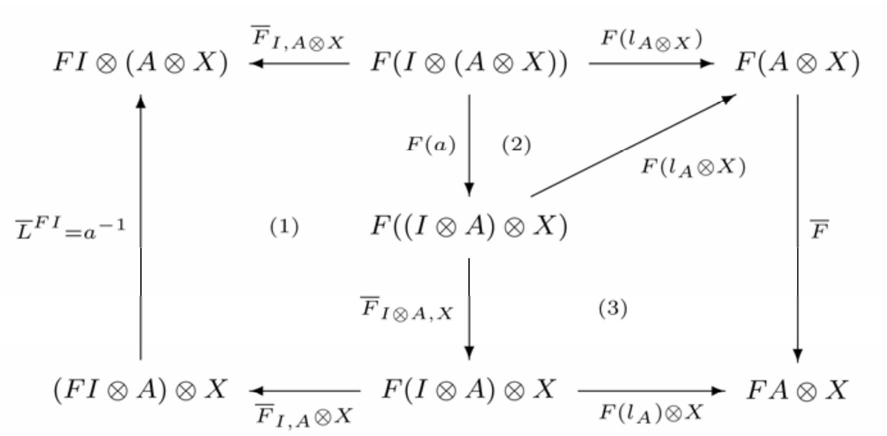
$$\Phi\Gamma(\alpha) = \Phi(\alpha_I) : (L^{FI}, \overline{L}^{FI}) \to (L^{GI}, \overline{L}^{GI}),$$

where $\alpha : (F, \overline{F}), (G, \overline{G})$.

We define the natural isomorphism $\rho : \Phi\Gamma \cong Id_{\mathbf{M(C)}}$ as follows:

$$\rho_{(F,\overline{F})}(X) = F(l_X)(\overline{F}_{I,X})^{-1} : (L^{FI}, \overline{L}^{FI})(X) \to (F, \overline{F})(X). \tag{3.6}$$

Consider the following diagram:



In this diagram, the region (1) commutes since it is the Diagram (3.1) for *M*-functor $(F, \overline{F})$, the region (2) commutes thanks to the compatibility of the associativity constraint a with the unit constraint (I, l, r) (image through *F*), and the region (3) commutes thanks to the naturality of $\overline{F}$. Hence, the perimeter which is the Diagram (3.3) for the *M*-morphism $\rho_{(F, \overline{F})}$ commutes. This follows that the definition of the natural isomorphism ρ is well-defined. Since Φ is a categorical equivalence and from Theorem 2.1, we infer that Φ is a monoidal equivalence.

3.2. Almost strict Ann-categories

An Ann-category **A** is called *almost strict*, if its constraints, except for a distributivity one (left or right) and the commutativity one, are all identities.

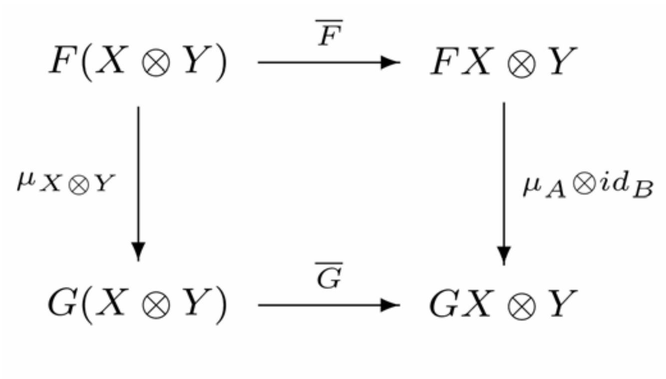
We now construct an almost strict Ann-category $\mu(\mathbf{A})$ based on the Ann-category **A**. First, let us assume that the Ann-category **A** is a strict monoidal category with respect to $\oplus$ (since any Ann-category is Ann-equivalent to a such Ann-category, Proposition 4.1 [4]).

Definition 3.7. The triple $(F, \breve{F}, \overline{F})$ is a μ-*functor*, if $(F, \breve{F})$ is a symmetric monoidal endo-equivalence with respect to $\oplus$, and $(F, \overline{F})$ is an *M*-functor with respect to $\otimes$ of the Ann-category **A**, such that the following conditions hold:

(i) the family $(\overline{F}_{X,-})$ is an $\oplus$- morphism from $F \circ L^X$ to L^{FX},

(ii) the family $(\overline{F}_{-,Y})$ is an $\otimes$- morphism from $F \circ R^Y$ to $L^Y \circ F$.

A μ-*morphism* from $(F, \breve{F}, \overline{F})$ to $(G, \breve{G}, \overline{G})$ is an $\oplus$- morphism $\phi : F \to G$ making the following diagram commute:



Example. For any object $A \in \mathbf{A}$, the pair $(L^A, \breve{L}^A)$, where $\breve{L}^A_{X,Y} = \mathfrak{L}_{A,X,Y}$ is a μ-functor. Any morphism $u : A \to B$ determines a μ- morphism $\phi : (L^A, \breve{L}^A, \overline{L}^A) \to (L^B, \breve{L}^B, \overline{L}^B)$ defined by $\phi_X = u \otimes^{id} X$.

Hereafter, we denote a sub-category of $\mathbf{M}(\mathbf{A})$ by $\mu(\mathbf{A})$, whose objects are μ- functors of $\mathbf{A}$ and whose morphisms are μ-morphisms. Then $\mu(\mathbf{A})$ is equipped with a strict monoidal structure induced by the one on $\mathbf{M}(\mathbf{A})$.

One can verify the following lemmas:

Lemma 3.8. $\mu(\mathrm{A})$ *is a category with the operation* $\oplus$ *defined by*

$$(F \oplus G)X = FX \oplus GX,$$

$$(F \breve{\oplus} G)_{X,Y} = \nu(\breve{F}_{X,Y} \oplus \breve{G}_{X,Y}),$$

$$(\phi \oplus \psi)_X = \phi_X \oplus \psi_X,$$

where $\nu = \nu_{A,B,C,D} : (A \oplus B) \oplus (C \oplus D) \to (A \oplus C) \oplus (B \oplus D)$ *is the morphism built uniquely from the constraints* a^{+}, id, c *in the Pic-category* $(\mathbf{A}, \oplus)$.

Lemma 3.9. $\mu(\mathrm{A})$ *is a category, whose the associativity and unit constraints are strict. Moreover, it has*

(i) *the zero object* $0^* = (\theta, \breve{\theta}, \overline{\theta})$ *given by*

$$\theta(X) = 0, \quad \theta(f) = id_0, \quad \breve{\theta}_{X,\,Y} = id_0, \quad \overline{\theta}_{X,\,Y} = (\widehat{R}^{Y})^{-1};$$

(ii) *the commutativity constraint* $c^*_{F,G}(X) = c_{FX,GX}$.

Proposition 3.10. $\mu(\mathrm{A})$ *is an almost Ann-category, whose the distributivity constraints given by*

$$\mathfrak{L}^*_{F,\,G,\,H}(X) = \breve{F}_{GX,\,HX}, \; \mathfrak{R}^* = Id.$$

We are now ready to prove the main result of this section. This result was introduced in [5] (in Vietnamese). Here, we shall give a full and exact proof thanks to the results in Subsection 3.1 and Theorem 2.2.

Theorem 3.11. *Any Ann-category is Ann-equivalent to an almost strict Ann-category.*

Proof. As mentioned above, we always can suppose that the category $\mathbf{A}$ is strict monoidal with respect to the operation $\oplus$. We now show that $\mathbf{A}$ and the almost strict Ann-category $\mu(A)$ are equivalent. Consider the functor $\Phi : \mathbf{A} \to \mu(\mathrm{A})$ given by

$$\Phi(A) = (L^A, \breve{L}^A, \overline{L}^A),$$

$$\Phi(u) = L(u) : L^A \to L^B, \quad L(u)_X = u \otimes X,$$

$$\breve{\Phi}_{A,\,B}(X) = \mathfrak{R}_{A,B,X}, \quad \widetilde{\Phi}_{A,B}(X) = (a_{A,B,X})^{-1}.$$

We shall prove that Φ is an equivalence. Consider the functor $J : \mu(\mathrm{A}) \to \mathbf{A}$ defined by

$$J\,(F,\ \breve{F},\ \overline{F}) = F\,(I), \quad J\,(\,F \xrightarrow{\phi} G\,) = (\,FI \xrightarrow{\phi_I} GI\,).$$

According to the proof of Theorem 3.6, we have isomorphisms

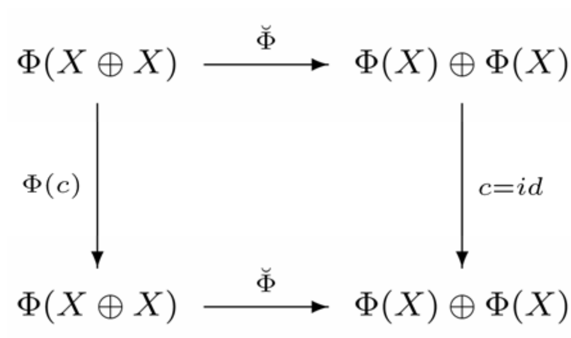
$$\alpha : J\Phi \cong Id_{\mathbf{A}}, \quad \beta : \Phi J \cong Id_{\mathbf{B}},$$

where $\alpha = r$, $\beta_F = \rho_F$ (ρ_F is defined by the relation (3.6)). One can verify that ρ_F is an $\oplus$- morphism, and therefore is a μ- morphism. This shows that Φ is an equivalence. According to Theorem 2.5, Φ is an Ann-equivalence. □

Proposition 3.12. *The condition* $c_{X,X} = id$ (*the regular condition*) *for any object* $X \in \mathbf{A}$ *is necessary and sufficient for the Ann-category* **A** *to be Ann-equivalent to an Ann-category*, *whose commutativity constraint is the identity*.

Proof. Assume that the Ann-category **A** satisfies the regular condition for the commutativity constraint *c*, in the sense $c_{X,X} = id$, for all $X \in \mathbf{A}$. Then **A** is Ann-equivalent to $\mathbf{A}'$, which is symmetric monoidal category with respect to $\oplus$. By Proposition 3.9, the commutativity constraint c^* of $\mu(\mathbf{A}')$ is the identity.

Conversely, from the commutative diagram



we have $\Phi(c_{X,X}) = Id$, where Φ is an Ann-equivalence. Therefore, $c_{X,X} = id$. □

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